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ORIGINAL PAPER

Axisymmetric problem for a spherical crack on the interface of elastic media

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Abstract A problem concerning a spherical interfacial crack is solved by the eigenfunction method. The problem is reduced to a coupled system of dual-series equations in terms of Legendre functions and then to a system of singular integral equations for two unknown functions. The behaviour of the solution near the edge of the spherical crack, and the stress-intensity factors and crack-opening displacements are studied. The case when the crack surfaces are under normal internal pressure of constant intensity is examined.

Keywords Cavity · Composite · Elastic · Inclusion · Interface spherical crack

1 Introduction

Aging and damage of materials are processes that are the causes of many dramatic events in the world. They may lead to catastrophic failures in oil and gas-storage tanks, pressure vessels, turbine-generator rotors, steam boilers, pipelines, bridges, airplanes, railways and welded ships [1]. It is also known that electronic chips sometimes become disfunctional due to mechanical damage. Scientific and engineering evidence shows that cracks in materials are the first steps in a sequence of processes leading to their fracture. Internal cracks, in the form of breaks in material solidity, have been examined in the literature for quite a long time [1, 2]. Special attention [3, 4] has been paid to linear crack problems in unbounded elastic bodies; the main results were obtained for flat penny-shaped or elliptic cracks. A comprehensive review of the state-of-the-art can be found in [2, 5, 6]. However, according to experimental analysis of the surfaces of damaged objects, the initial surfaces of the material breaks are not flat, being mainly of spherical or ellipsoidal shape [5, 7]. To evaluate the strength of a material with internal cracks, one can start from the solution of a class of problems within the theory of elasticity for three-dimensional bodies weakened by cracks with curved surfaces. Such cracks could be modelled by cuts on a part of some surface of revolution having non-zero curvature. In this case, one has the possibility to vary the geometrical parameters of the

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surface and, by doing this, to make these look more like real cracks. An experimental study of composite materials emphasizes the practical significance of this class of problems. Specifically, heterogeneous media are being filled with particles of spherical or ellipsoidal form (for example, a wolframium-carbide matrix reinforced by diamond grains) [8, 9]. Their mechanical characteristics depend, to a considerable degree, on the material solidity break, which appears, as a rule, on the interphase boundary and is located on parts of the spherical or ellipsoidal surface. The theory of non-flat cracks presents quite natural tendencies, when a complicated geometry of cuts prompts the need to develop more complex mathematical methods and to increase significantly the number of mathematical operations needed for their analysis. This provides an explanation of the fact that only a few publications deal with studies of stress fields and displacements of elastic bodies with non-flat cuts, although the importance of such studies was repeatedly emphasized in the literature [10].

A problem concerning a spherical crack in an elastic solid was the first in a sequence of problems about non-flat cracks. The earliest works [11, 12] used representations in terms of analytic functions of a complex variable. Later these results appeared to be incorrect [13]. Subsequent papers [14, 15] proposed a new approach based on an eigenfunction method. The problem was reduced to a coupled system of dual-series equations in terms of Legendre functions. A specific choice of integral operators for the auxiliary functions allowed one to reduce the problem, either to a system of integro-differential equations [15] or to a system of Fredholm integral equations of the second kind [14]. Reference [14] used a collocation method to solve numerically the system of Fredholm integral equations. The stress distributions near the spherical surface were accurately estimated. In [15] a general solution of the problem in question in explicit analytical form was suggested, along with an analytical method to solve some integro-differential equations. A local analysis of stresses and displacements near the edge of the spherical crack was performed and the stress-intensity factors (SIF) were obtained.

These methods were critically reviewed by Martin [16] and he suggested to use the method by Martynenko and Ulitko as the simplest. He essentially developed this method and provided an accurate asymptotic analysis for a shallow spherical-cap crack in a homogenous elastic space.

Later both teams (Martynenko, Ulitko (1982, 1983) and Altenbach, Smirnov et al. (1986, 1995)) applied their methods to the problem concerning a spherical interfacial crack. Altenbach et al. [17] solved the problem numerically and calculated the SIF. Martynenko and Ulitko [18, 19] obtained analytical results for the SIF and crack-opening displacements. Although very effective, the Martynenko and Ulitko method was not exposed sufficiently in the scientific literature. The purpose of this paper is to describe this method in full detail and to revisit the problem of a spherical interfacial crack.

2 Statement of the problem

Let us consider (Fig. 1) an elastic space (ν_2, G_2) with a partially separated spherical inclusion (ν_1, G_1) in an axisymmetric field of external forces extending to infinity. Here (ν_2, G_2) and (ν_1, G_1) are the Poisson ratio and the shear modulus of the elastic space and the inclusion, respectively. The crack is modelled by a mathematical cut AMB on a part of spherical surface $(r = r_0, 0 \le \theta \le \theta_0, 0 \le \varphi \le 2\pi)$. It is assumed that the surfaces of the crack do not enter into contact interaction, and that the connection on the remaining part of the boundary is ideal.

To solve this problem, let us use the superposition principle (Fig. 2). Let us look for the solution of problem A as a sum of the solutions of the two following static problems: problem B concerns the stress field of the entire elastic body affected by forces and problem C concerns the equilibrium of an elastic space with a cut on the surface S, when the forces are applied only to the surface of the crack. In problem B the system of external forces coincides with the external forces of problem A, while in problem C the forces on the surface of the cut are equal in magnitude and opposite in direction to those forces that occur on the conventionally chosen surface of the crack S in problem B. The superposition of the solution of

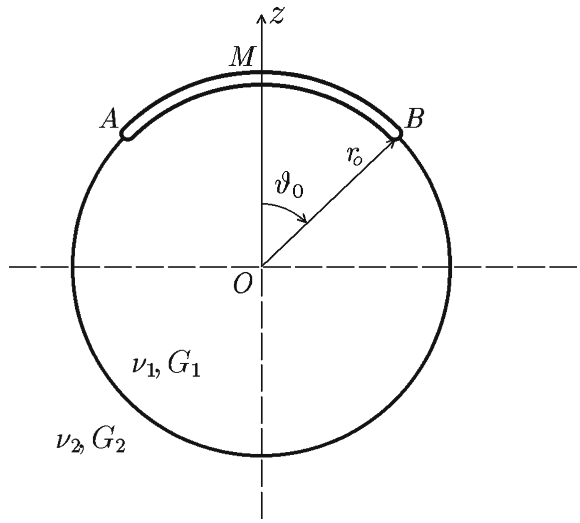

Fig. 1 A spherical-cap crack on the interface boundary of a spherical inclusion and an elastic space

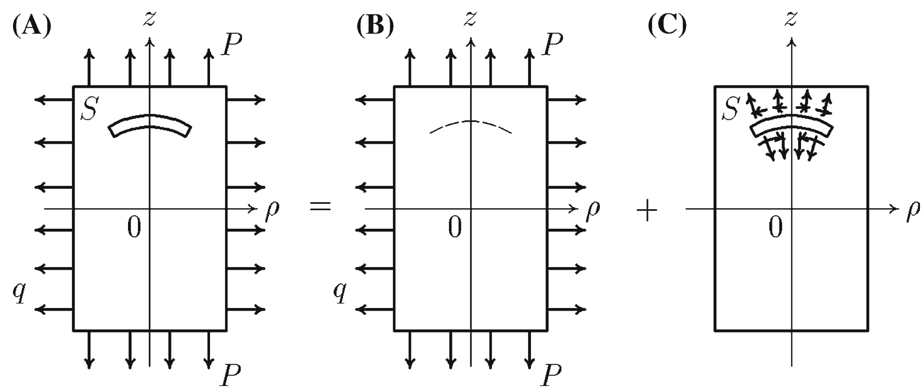

Fig. 2 The superposition principle: the solution of problem A is the sum of the solutions of the two static problems: problem B on the stress-strained stage of the entire elastic body affected by forces and problem C on the equilibrium of an elastic space with a cut on the surface S, when the forces are applied only to the surface of the crack. In the problem B the system of external forces coincides with the external forces of problem A, while in problem C the forces on the surface of the cut are equal in magnitude and opposite in direction to those forces that occur on the conventionally chosen surface of the crack S in problem B.

problems B and C will give a solution to the problem regarding the equilibrium of the elastic surface with the cut, provided the surfaces of the crack are stressless and the body is influenced by the external field of the axisymmetric forces only. In problems A, B, C the body must have the same geometry and the same mechanical characteristics. This body may be both homogeneous and inhomogeneous. The solution to problem B is relatively simple and the stress field does not have singularities in the internal points. The character of the singularities of the stresses in the initial problem will result from the solution of problem C. Further research will focus on problem C with forces applied on the crack's surface [20].

The next stage of the definition of the problem is based on the use of the method of partial areas: the complex area is divided into two or more simpler areas in such a way that the physical fields in each of them will be given in the form of eigenfunction expansions. Conditions of continuity of the physical fields on common sections of the neighbouring areas have to be satisfied.

Let us divide the composite elastic space into two domains: an inner sphere ($r \le r_0$) and an external sphere ($r \ge r_0$) (Fig. 1). For each domain the Lamé vector equation of equilibrium,

$$2\frac{1-\nu}{1-2\nu}\operatorname{grad}\operatorname{div}\vec{u} - \operatorname{rot}\operatorname{rot}\vec{u} = 0, \tag{1}$$

must be solved. The unknown coefficients are found from the boundary conditions

$$\begin{aligned}&\sigma_r^{(1)}=\sigma_r^{(2)},\quad \sigma_{r\theta}^{(1)}=\sigma_{r\theta}^{(2)},\quad u_r^{(1)}=u_r^{(2)},\quad u_\theta^{(1)}=u_\theta^{(2)},\quad (r=r_0,\theta_0\le\theta\le\pi),\\&\sigma_r^{(1)}=\sigma_r^{(2)}=f_1(\theta),\quad \sigma_{r\theta}^{(1)}=\sigma_{r\theta}^{(2)}=f_2(\theta),\quad (r=r_0,0\le\theta<\theta_0),\end{aligned}\tag{2}$$

where $f_i(\theta)$ are given functions equal to the loading transferred to the surface of the cut according to the superposition principle; $\sigma_r^{(i)},\sigma_{r\theta}^{(i)},u_r^{(i)},u_\theta^{(i)}$ are the components of the stress tensor and the displacement vector for space ($i=2$) and spherical inclusion ($i=1$), ν_i are the Poisson ratios, G_i are the shear moduli of space ($i=2$) and inclusion ($i=1$) materials.

3 Method of solution

In mathematical terms, the solution of the problem proceeds as follows: the displacement-vector components and the stress-tensor components are represented by eigenfunction expansions. The result is a coupled system of dual-series equations for infinite sequences of coefficients.

For the initial relationships, let us take the general solutions of elastic problems concerning a spherical inclusion and space with a spherical cavity represented in the form of series expansions of the Legendre function [21]

$$\begin{aligned}2G_1u_r^{(1)}&=\sum_{n=0}^{\infty}\left[(n-2+4\nu_1)A_nr^{n+1}+B_nr^{n-1}\right]P_n(\cos\theta),\\2G_1u_\theta^{(1)}&=\sum_{n=1}^{\infty}\left[n(n+5-4\nu_1)A_nr^{n+1}+(n+1)B_nr^{n-1}\right]\frac{P_n^1(\cos\theta)}{n(n+1)},\\\sigma_r^{(1)}&=\sum_{n=0}^{\infty}\left[(n^2-n-2-2\nu_1)A_nr^n+(n-1)B_nr^{n-2}\right]P_n(\cos\theta),\\\sigma_{r\theta}^{(1)}&=\sum_{n=1}^{\infty}\left[n(n^2+2n-1+2\nu_1)A_nr^n+(n^2-1)B_nr^{n-2}\right]\frac{P_n^1(\cos\theta)}{n(n+1)},\end{aligned}\tag{3}$$

$$\begin{aligned}2G_2u_r^{(2)}&=\sum_{n=0}^{\infty}\left[(n+3-4\nu_2)C_nr^{-n}-D_nr^{-n-2}\right]P_n(\cos\theta),\\2G_2u_\theta^{(2)}&=\sum_{n=1}^{\infty}\left[-(n+1)(n-4+4\nu_2)C_nr^{-n}+nD_nr^{-n-2}\right]\frac{P_n^1(\cos\theta)}{n(n+1)},\\\sigma_r^{(2)}&=\sum_{n=0}^{\infty}\left[-(n^2+3n-2\nu_2)C_nr^{-n-1}+(n+2)D_nr^{-n-3}\right]P_n(\cos\theta),\\\sigma_{r\theta}^{(2)}&=\sum_{n=1}^{\infty}\left[(n+1)(n^2-2+2\nu_2)C_nr^{-n-1}-n(n+2)D_nr^{-n-3}\right]\frac{P_n^1(\cos\theta)}{n(n+1)}.\end{aligned}\tag{4}$$

Here A_n,B_n $(B_0=0),C_n,D_n$ are infinite sequences of dimensionless coefficients. From the boundary conditions (2) we have

$$\sigma_r^{(1)}=\sigma_r^{(2)},\quad \sigma_{r\theta}^{(1)}=\sigma_{r\theta}^{(2)},\qquad (r=r_0,\ 0\le\theta\le\pi).$$

These conditions and the orthogonality of the Legendre functions on the interval $[0,\pi]$ yield an algebraic system of equations for the dimensionless coefficients. Solving it we have

$$
\begin{aligned}
C_n r_0^{-n-1} &= \Delta_{n2}^{-1}\left[n(n-1)(2n+3)A_n r_0^n + (n-1)(2n+1)B_n r_0^{n-2}\right],\\
D_n r_0^{-n-3} &= \Delta_{n2}^{-1}\left\{\left[2n^5+5n^4-5n^2+4(n^2+n+1)(\nu_1-\nu_2)-4(2n+1)\nu_1\nu_2+6n+4\right]A_n r_0^n\right.\\
&\quad\left.+(n^2-1)(n+2)(2n-1)B_n r_0^{n-2}\right\},\\
\Delta_{n2} &= 2\left[(2n+1)\nu_2-n^2-n-1\right].
\end{aligned}
\tag{5}
$$

Satisfying the boundary condition (2), we get a coupled system of dual-series equations

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{(n+2)\,\Delta_{n2}}\left(\alpha_{11}A'_n+\alpha_{12}B'_n\right)P_n(\cos\theta)=0,\quad (\theta_0\le\theta\le\pi),\\
&\sum_{n=1}^{\infty}\frac{1}{(n+2)\,\Delta_{n2}}\left(\alpha_{21}A'_n+\alpha_{22}B'_n\right)\frac{P_n^1(\cos\theta)}{n\,(n+1)}=0,\\
&\sum_{n=0}^{\infty}\left[\left(n^2-n-2-2\nu_1\right)A'_n+(n-1)\,B'_n\right]P_n(\cos\theta)=f_1(\theta),\quad \left(0\le\theta<\theta_0\right),\\
&\sum_{n=1}^{\infty}\left[n\left(n^2+2n-1+2\nu_1\right)A'_n+\left(n^2-1\right)B'_n\right]\frac{P_n^1\left(\cos\theta\right)}{n\left(n+1\right)}=f_2(\theta),
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
\alpha_{11} &= \frac{1}{2}G_2\,(n+2)\,(n-2+4\nu_1)\,\Delta_{n2}-G_1\Big[3n^4+7n^3-2n^2-12n-2+2\left(\nu_2-\nu_1\right)\left(n^2+n+1\right)+\\
&\quad+2\nu_1\nu_2\left(2n+1\right)-2n\nu_2\left(2n^3+5n^2-n-6\right)\Big],\\
\alpha_{12} &= (n+2)\left\{\frac{1}{2}G_2\Delta_{n2}-G_1\,(n-1)\left[3n+2-2\nu_2\,(2n+1)\right]\right\},\\
\alpha_{21} &= n\Big\{\frac{1}{2}G_2\,(n+2)\,(n+5-4\nu_1)\Delta_{n2}-G_1\big[3n^4+12n^3+9n^2-8n-10-2\nu_2(n^2-1)\\
&\quad\times(n+2)\,(n+5-4\nu_1)-2(\nu_2-\nu_1)(n^2+n+1)-2\nu_1\nu_2(2n+1)\big]\Big\},\\
\alpha_{22} &= (n+1)\,\alpha_{12};\quad A'_n=A_n r_0^n;\quad B'_n=B_n r_0^{n-2}.
\end{aligned}
\tag{7}
$$

The solution to the system (6) will be in the form of integral operators [15, 22]

$$
\begin{aligned}
\frac{1}{(n+2)\,\Delta_{n2}}\left(\alpha_{11}A'_n+\alpha_{12}B'_n\right) &= 2G_1\int_0^{\theta_0}\varphi(t)\sin\left(n+\frac{1}{2}\right)t\mathrm{d}t=I_n^{(1)},\\
\frac{(2n+1)^{-1}}{(n+2)\,\Delta_{n2}}\left(\alpha_{21}A'_n+\alpha_{22}B'_n\right) &= G_1\int_0^{\theta_0}\psi(t)\cos\left(n+\frac{1}{2}\right)t\mathrm{d}t=I_n^{(2)},
\end{aligned}
\tag{8}
$$

where the introduced auxiliary functions $\varphi(t)$ and $\psi(t)$ are assumed to be continuous for $t\in[0,\theta_0)$ and have the following properties

$$
\varphi(-t)=-\varphi(t),\quad \psi(-t)=\psi(t),\int_0^{\theta_0}\psi(t)\cos\frac{t}{2}\mathrm{d}t=0.
\tag{9}
$$

By solving the systems (8), (5) step-by-step, we get

$$C_n r_0^{-n-1} = C'_n = \frac{n-1}{\alpha_{13}}\left[nI_n^{(1)} + (2n+1)\,I_n^{(2)}\right],$$
$$D_n r_0^{-n-3} = D'_n = \frac{1}{\alpha_{13}\alpha_{31}}\left[\beta_{21}I_n^{(1)} + \beta_{22}\,(2n+1)\,I_n^{(2)}\right], \tag{10}$$

where

$$\alpha_{13} = \frac{1}{2}G_2\Delta_{n2} - G_1\,(n-1)\left[3n+2-2\nu_2\,(2n+1)\right],$$
$$\alpha_{31} = G_2\,(n+2)\left[2\nu_1\,(2n+1) - 3n - 1\right] - \frac{1}{2}G_1\Delta_{n1},$$
$$\beta_{21} = (n+1)\Bigg\{\frac{1}{2}G_1(n-1)\Delta_{n1}(n-4+4\nu_2) + G_2\big[3n^4 - 9n^2 + 2n - 2$$
$$+(\nu_2-\nu_1)(n^2+n+1) + 2\nu_1\nu_2(2n+1) - 2\nu_1 n(n-1)(2n-1)(n+2)\big]\Bigg\}. \tag{11}$$
$$\beta_{22} = 2\Big[3n^4 + 5n^3 - 5n^2 - n + 4 - 2\nu_1\left(n^2-1\right)(2n-1)\,(n+2)$$
$$-2\,(\nu_2-\nu_1)\left(n^2+n+1\right)\Big] + \frac{1}{2}G_1\,(n-1)\,(n+3-4\nu_2)\,\Delta_{n1},$$
$$\Delta_{n1} = 2\left[n^2+n+1+(2n+1)\,\nu_1\right].$$

Upon the integration of the last two equations of system (6) within the limits from 0 to θ and substitution for the unknown constant expressions (8), we get the following equations

$$\sum_{n=0}^{\infty}\big(L_{11}(n)I_n^{(1)} + L_{12}(n)I_n^{(2)}\big)P_n\big(\cos\theta\big) = f_1\big(\theta\big),$$
$$\sum_{n=1}^{\infty}\big(L_{21}(n)I_n^{(1)} + L_{22}(n)I_n^{(2)}\big)\frac{P_n^1\big(\cos\theta\big)}{n\big(n+1\big)} = f_2\big(\theta\big), \quad \big(0 \le \theta < \theta_0\big), \tag{12}$$

where

$$L_{11}(n) = \frac{2}{\alpha_{12}\Delta_1}\big\{G_1(n-1)\big[2n^4+7n^3+11n^2+9n+4+\nu_1(2n+1)(2n^2+5n+4)$$
$$-\nu_2(n^2+n+1)(2n^2+7n+4) - \nu_1\nu_2(2n+1)(2n^2+7n+4)\big]$$
$$+G_2(n+2)\big[n^2+n+1-\nu_2(2n+1)\big]\big[2n^2-n+1-\nu_1(2n^2-3n-1)\big]\big\},$$

$$L_{12}(n) = \frac{2(2n+1)}{\Delta_1\alpha_{12}}\big\{G_1(n-1)\big[\nu_2(2n+3)(n^2+n+1) - n^3 - 4n^2 - 4n - 3 - \nu_1(2n^2+7n+3)$$
$$+\nu_1\nu_2(2n+1)(2n+3)\big] - \frac{1}{2}G_2(n+2)\Delta_{n2}\big[n-2-(2n-1)\nu_1\big]\big\}, \tag{13}$$

$$L_{22}(n) = \frac{2(2n+1)}{\Delta_1\alpha_{12}}\big\{G_1(n-1)\big[\nu_1(4n^3+10n^2+6n+1) + 2n^4+6n^3+7n^2+5n+1$$
$$-\nu_2(n^2+n+1)(2n^2+5n+1) - \nu_1\nu_2(2n+1)(2n^2+5n+1)\big]$$
$$-\frac{1}{2}G_2(n+2)\Delta_{n2}\big[2n^2-1-(2n^2-n-2)\nu_1\big]\big\},$$

$$L_{21}(n) = \frac{n\,(n+1)}{2n+1}L_{12}(n), \quad \Delta_1 = G_2[2\nu_1(2n+1)-3n-1]$$
$$-G_1(n+2)[n^2+n+1+\nu_1(2n+1)].$$

If we use the following integral representations for the Legendre functions [22, 23]

$$
\begin{aligned}
&P_n(\cos\theta) = \frac{2}{\pi}\int_0^{\theta} \frac{\cos(n+\frac{1}{2})x}{\sqrt{2\cos x - 2\cos\theta}}\,\mathrm{d}x,\\
&\frac{n+\frac{1}{2}}{n(n+1)}P_n^1(\cos\theta) = -\frac{2}{\pi}\int_0^{\theta} \frac{\sin x\sin(n+\frac{1}{2})x}{\sin\theta\sqrt{2\cos x - 2\cos\theta}}\,\mathrm{d}x,\\
&\frac{n+\frac{1}{2}}{n(n+1)}P_n^1(\cos\theta) = \frac{2}{\pi}\int_{\theta}^{\pi} \frac{\sin x\cos(n+\frac{1}{2})x}{\sin\theta\sqrt{2\cos\theta - 2\cos x}}\,\mathrm{d}x
\end{aligned}
\tag{14}
$$

and change in (12) the order of integration and summation, we arrive at a system of Abel integral equations. Having solved it and subsequently integrated the resulting correlations with respect to the variable x, we obtain the following system of functional equations

$$
\begin{aligned}
&\frac{2}{\pi}\sum_{n=0}^{\infty}\left[L_{11}(n)I_n^{(1)} + L_{12}(n)I_n^{(2)}\right]\frac{\sin\left(n+\frac{1}{2}\right)x}{n+\frac{1}{2}} = F_1(x),\\
&\frac{2}{\pi}\sum_{n=1}^{\infty}\left[L_{21}(n)I_n^{(1)} + L_{22}(n)I_n^{(2)}\right]\frac{\cos\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)^2} = F_2(x) + c,
\end{aligned}
\tag{15}
$$

where c is an unknown constant and $F_i(x)$ are, respectively, given by

$$
\begin{aligned}
&F_1(x) = \frac{2}{\pi}\int_0^{x}\frac{f_1(\theta)\sin\theta\,\mathrm{d}\theta}{\sqrt{2\cos\theta - 2\cos x}},\\
&F_2(x) = \frac{2}{\pi}\int_0^{x}\frac{1}{\sin x}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^{x}\frac{f_2(\theta)\sin^2\theta\,\mathrm{d}\theta}{\sqrt{2\cos\theta - 2\cos x}}\right]\right\}\mathrm{d}x.
\end{aligned}
\tag{16}
$$

Because of the symmetry of the problem and the properties (9) of the functions $\varphi(t), \psi(t)$, the equalities (15) could be formulated as follows:

$$
\begin{aligned}
&\int_{-\theta_0}^{\theta_0}\left[\varphi(t)M_{11}(t,x) + \psi(t)M_{12}(t,x)\right]\mathrm{d}t = F_1(x),\\
&\int_{-\theta_0}^{\theta_0}\left[\varphi(t)M_{21}(t,x) + \psi(t)M_{22}(t,x)\right]\mathrm{d}t = F_2(x),
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
&M_{11}(t,x) = \frac{2}{\pi}\sum_{n=0}^{\infty}L_{11}(n)\left(n+\frac{1}{2}\right)^{-1}\sin\left(n+\frac{1}{2}\right)t\sin\left(n+\frac{1}{2}\right)x,\\
&M_{12}(t,x) = \frac{1}{\pi}\sum_{n=0}^{\infty}L_{12}(n)\left(n+\frac{1}{2}\right)^{-1}\cos\left(n+\frac{1}{2}\right)t\sin\left(n+\frac{1}{2}\right)x,\\
&M_{21}(t,x) = \frac{2}{\pi}\sum_{n=0}^{\infty}L_{21}(n)\left(n+\frac{1}{2}\right)^{-2}\sin\left(n+\frac{1}{2}\right)t\cos\left(n+\frac{1}{2}\right)x,\\
&M_{22}(t,x) = \frac{1}{\pi}\sum_{n=0}^{\infty}L_{22}(n)\left(n+\frac{1}{2}\right)^{-2}\cos\left(n+\frac{1}{2}\right)t\cos\left(n+\frac{1}{2}\right)x.
\end{aligned}
$$

Separation of the main components in the expressions (13) for L_{ij} and use of the following sums in the class of generalized functions

$$\begin{aligned}
&\sum_{n=0}^{\infty}\cos\left(n+\frac{1}{2}\right)t\cos\left(n+\frac{1}{2}\right)x=\frac{\pi}{2}\delta(t-x),\quad (0<t,x<\pi),\\
&\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\sin\left(n+\frac{1}{2}\right)t\cos\left(n+\frac{1}{2}\right)x=-\frac{\pi}{2}\delta_t'(t-x),\\
&\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)t\sin\left(n+\frac{1}{2}\right)x=\frac{\pi}{2}\delta_t'(t-x),\\
&\sum_{n=0}^{\infty}\sin\left(n+\frac{1}{2}\right)t\sin\left(n+\frac{1}{2}\right)x=\frac{\pi}{2}\delta(t-x),\\
&\sum_{n=0}^{\infty}\cos\left(n+\frac{1}{2}\right)t\sin\left(n+\frac{1}{2}\right)x=\frac{1}{4}\left[\operatorname{cosec}\frac{x-t}{2}+\operatorname{cosec}\frac{x+t}{2}\right],\\
&\sum_{n=0}^{\infty}\frac{\sin(n+\frac{1}{2})t\sin(n+\frac{1}{2})x}{n+\frac{1}{2}}=\frac{1}{2}\log\left|\frac{\cot\frac{t-x}{4}}{\cot\frac{t+x}{4}}\right|,\\
&\sum_{n=0}^{\infty}\frac{\cos(n+\frac{1}{2})t\cos(n+\frac{1}{2})x}{n+\frac{1}{2}}=\frac{1}{2}\log\left|\cot\frac{t-x}{4}\cot\frac{t+x}{4}\right|
\end{aligned}\tag{19}$$

leads to a system of singular integral equations

$$\begin{aligned}
&\varphi(x)-\frac{b_0}{2\pi a_0}\int_{-\theta_0}^{\theta_0}\frac{\psi(t)\,\mathrm{d}t}{\sin\frac{t-x}{2}}+\int_{-\theta_0}^{\theta_0}\Big[K_{11}(t,x)\varphi(t)+K_{12}(t,x)\psi(t)\Big]\mathrm{d}t=a_0^{-1}F_1(x),\\
&\psi(x)+\frac{b_0}{2\pi a_0}\int_{-\theta_0}^{\theta_0}\frac{\varphi(t)\,\mathrm{d}t}{\sin\frac{t-x}{2}}+\int_{-\theta_0}^{\theta_0}\Big[K_{21}(t,x)\varphi(t)+K_{22}(t,x)\psi(t)\Big]\mathrm{d}t=a_0^{-1}F_2(x).
\end{aligned}\tag{20}$$

Here

$$\begin{aligned}
&K_{11}(t,x)=\frac{a_1}{\pi a_0}\left\{\log\left|\cot\frac{t-x}{4}\right|-\log\left|\cot\frac{t+x}{4}\right|\right\}+\frac{2}{\pi a_0}\sum_{n=0}^{\infty}\overline{L}_{11}(n)\sin\left(n+\frac{1}{2}\right)t\sin\left(n+\frac{1}{2}\right)x,\\
&K_{12}(t,x)=\frac{b_1}{4a_0}H(x-t)+\frac{1}{\pi a_0}\sum_{n=0}^{\infty}\overline{L}_{12}(n)\cos\left(n+\frac{1}{2}\right)t\sin\left(n+\frac{1}{2}\right)x,\\
&K_{21}(t,x)=\frac{b_1}{4a_0}H(t-x)+\frac{2}{\pi a_0}\sum_{n=0}^{\infty}\overline{L}_{21}(n)\sin\left(n+\frac{1}{2}\right)t\cos\left(n+\frac{1}{2}\right)x,\\
&K_{22}(t,x)=\frac{c_1}{2\pi a_0}\log\left|\frac{\cot\frac{t-x}{4}}{\cot\frac{t+x}{4}}\right|+\frac{1}{\pi a_0}\sum_{n=0}^{\infty}\overline{L}_{22}(n)\cos\left(n+\frac{1}{2}\right)t\cos\left(n+\frac{1}{2}\right)x,
\end{aligned}\tag{21}$$

$$\begin{aligned}
&\overline{L}_{11}(n)=L_{11}(n)\left(n+\frac{1}{2}\right)^{-1}-a_0-a_1\left(n+\frac{1}{2}\right)^{-1},\\
&\overline{L}_{12}(n)=L_{12}(n)\left(n+\frac{1}{2}\right)^{-1}-b_0-b_1\left(n+\frac{1}{2}\right)^{-1},
\end{aligned}\tag{22}$$

$$\overline{L}_{21}(n) = L_{21}(n)\left(n+\frac{1}{2}\right)^{-2} - \frac{1}{2}b_0 - \frac{1}{2}b_1\left(n+\frac{1}{2}\right)^{-1},$$

$$\overline{L}_{22}(n) = L_{22}(n)\left(n+\frac{1}{2}\right)^{-2} - 2a_0 - c_1\left(n+\frac{1}{2}\right)^{-1},$$

$$a=\frac{b_0}{a_0},\quad a_0=\frac{a_2}{a_3},\quad a_1=\frac{a_4a_3-a_2a_5}{a_3^2}, b_0=\frac{b_2}{a_3},\quad b_1=\frac{b_3a_3-b_2a_5}{a_3^2},\quad c_1=\frac{c_2d_3-2a_2a_5}{a_3^2},$$

$$\begin{aligned}
&a_2 = 4G_1[G_1(1-\nu_2)+G_2(1-\nu_1)],\\
&b_2 = 4G_1[G_2(1-\nu_1)+G_1(1-\nu_2)],\\
&a_3 = [(4\nu_1-3)G_2-G_1][(4\nu_2-3)G_1-G_2],\\
&a_4 = 4G_1[G_2(2\nu_1-\nu_2-2\nu_1\nu_2)+G_2(\nu_1-2\nu_2-2\nu_1\nu_2)],\\
&a_5 = 2[(3\nu_1-2)G_2-\nu_1G_1][(4\nu_2-3)G_1-G_2]-2[(3\nu_2-2)G_1-\nu_2G_2][(4\nu_1-3)G_2-G_1],\\
&b_3 = 4G_1[G_1(4\nu_1\nu_2-2\nu_1-2\nu_2-1)+G_2(4\nu_1\nu_2-2\nu_2-\nu_1-1)],\\
&c_2 = 4G_1[G_1(1+4\nu_1-4\nu_1\nu_2)+G_2(1-4\nu_2+4\nu_1\nu_2)].
\end{aligned}\tag{23}$$

As is evident from (21), K_{12}, K_{21} are Fredholm kernels, and K_{11}, K_{22} have a logarithmic singularity. To determine the singularity of the solution of system (20), we shall regularize it [24]. Let us present (20) in the form of the complex equation

$$K_0f(x)+kf(x)=\Phi(x),\tag{24}$$

where

$$\Phi(x)=a_0^{-1}[F_2(x)+\mathrm{i}F_1(x)],\quad f(x)=\psi(x)+\mathrm{i}\varphi(x),$$

$$\bar f(x)=\psi(x)-\mathrm{i}\varphi(x),\quad K_0f(x)=f(x)+\frac{\gamma}{\pi\mathrm{i}}\int_{-\theta_0}^{\theta_0}\frac{f(t)\,\mathrm{d}t}{\sin\frac{t-x}{2}},$$

$$kf(x)=\int_{-\theta_0}^{\theta_0}[K_1(t,x)f(t)+K_2(t,x)\bar f(t)]\mathrm{d}t,\quad \gamma=\frac{b_0}{2a_0},\tag{25}$$

$$K_1(t,x)=\frac{1}{2}\left[K_{22}(t,x)+K_{21}(t,x)+\mathrm{i}\Big(K_{11}(t,x)+K_{12}(t,x)\Big)\right],$$

$$K_2(t,x)=\frac{1}{2}\left[K_{22}(t,x)-K_{21}(t,x)+\mathrm{i}\Big(K_{12}(t,x)-K_{11}(t,x)\Big)\right].$$

Based on the ideas and results described in [24–26], let us construct the solution of the characteristic equation

$$f(x)+\frac{\gamma}{\pi\mathrm{i}}\int_{-\theta_0}^{\theta_0}\frac{f(t)\,\mathrm{d}t}{\sin\frac{t-x}{2}}=F(x).\tag{26}$$

We consider the function of the complex variable z in the plane L, which has a cut along the line $[-\theta_0,\theta_0]$

$$\Psi(z)=\frac{1}{4\pi\mathrm{i}}\int_{-\theta_0}^{\theta_0}\frac{f(t)\,\mathrm{d}t}{\sin\frac{t-z}{2}}.\tag{27}$$

To determine this function is tantamount to solving the Riemann boundary-value problem [26]

$$\Psi^+=G\Psi^-+g(t),\tag{28}$$

where

$$\Psi^{+}(x)=\frac{1}{2}f(x)+\frac{1}{4\pi \mathrm{i}}\int\limits_{-\theta_0}^{\theta_0} f(t)\mathrm{cosec}\frac{t-x}{2}\mathrm{d}t,$$

$$\Psi^{-}(x)=-\frac{1}{2}f(x)+\frac{1}{4\pi \mathrm{i}}\int\limits_{-\theta_0}^{\theta_0} f(t)\mathrm{cosec}\frac{t-x}{2}\mathrm{d}t, \tag{29}$$

$$G=\frac{1-2\gamma}{1+2\gamma},\quad g(t)=\frac{F(t)}{1+2\gamma}.$$

Having solved the Riemann problem, we can represent the characteristic equation as

$$f(x)=\frac{Z(x)}{1-4\gamma^2}\left[F(x)-\frac{\gamma}{\pi \mathrm{i}}\int\limits_{-\theta_0}^{\theta_0}\frac{F(t)-F(x)}{Z(t)\sin\frac{t-x}{2}}\mathrm{d}t\right], \tag{30}$$

where

$$Z(x)=\sqrt{1-4\gamma^2}\left(\tan\frac{\theta_0-x}{4}\right)^{\mathrm{i}\lambda}\left(\tan\frac{\theta_0+x}{4}\right)^{-\mathrm{i}\lambda},\quad \lambda=\frac{1}{2\pi}\log\frac{1+2\gamma}{1-2\gamma}. \tag{31}$$

This solution of the characteristic equation (26) allows us to convert the singular equation (24) into a Fredholm equation, similar to the regularization of the singular equations as described in [24].

Let us rewrite Eq. (24) as follows

$$f(x)+\frac{\gamma}{\pi \mathrm{i}}\int\limits_{-\theta_0}^{\theta_0}\frac{f(t)\,\mathrm{d}t}{\sin\frac{t-x}{2}}=\Phi(x)-kf(x). \tag{32}$$

We temporarily assume that the right part of Eq. (32) is a known function. It is easy to show that the function $kf(x)$ is regular. Solving Eq. (32) by applying (21), we get

$$f(x)+\frac{Z(x)}{1-4\gamma^2}\left[kf(x)-\frac{\gamma}{\pi \mathrm{i}}\int\limits_{-\theta_0}^{\theta_0}\frac{kf(t)-kf(x)}{Z(t)\sin\frac{t-x}{2}}\mathrm{d}t\right]=\frac{Z(x)}{1-4\gamma^2}\left[\Phi(x)-\frac{\gamma}{\pi \mathrm{i}}\int\limits_{-\theta_0}^{\theta_0}\frac{\Phi(t)-\Phi(x)}{Z(t)\sin\frac{t-x}{2}}\mathrm{d}t\right]. \tag{33}$$

Thus, the singularity of the solution of Eq. (24) at the end of the interval $[-\theta_0,\theta_0]$ is determined by the function $Z(t)$. Hence, $f(t)$ can be represented in the form

$$f(t)=Z(t)\,L(t),\quad L(t)=L_1(t)+\mathrm{i}L_2(t),\quad L(-t)=\bar{L}(t), \tag{34}$$

where $L(t)$ does not have singularities at the ends of the interval $[-\theta_0,\theta_0]$.

By introducing

$$\frac{1}{1-4\gamma^2}\left[\Phi(x)-\frac{\gamma}{\pi \mathrm{i}}\int\limits_{-\theta_0}^{\theta_0}\frac{\Phi(t)-\Phi(x)}{Z(t)\sin\frac{t-x}{2}}\mathrm{d}t\right]=F_*(x), \tag{35}$$

we may easily show that Eq. (33) is equivalent to the Fredholm system

$$L(x)+\frac{1}{1-4\gamma^2}\int\limits_{-\theta_0}^{\theta_0}\left[L(t)\,M_1(t,x)+\bar{L}(t)\,M_2(t,x)\right]\mathrm{d}t=F_*(x), \tag{36}$$

$$M_1(t,x)=\frac{\mathrm{i}\gamma Z(t)}{\pi\left(1-4\gamma^2\right)}\int\limits_{-\theta_0}^{\theta_0}\bar{Z}(\tau)\,\frac{K_1(t,\tau)-K_1(t,x)}{\sin\frac{\tau-x}{2}}\mathrm{d}\tau+Z(t)\,K_1(t,x), \tag{37}$$

$$M_2(t,x) = \frac{\mathrm{i}\gamma \bar{Z}(t)}{\pi\left(1-4\gamma^2\right)} \int_{-\theta_0}^{\theta_0} \bar{Z}(\tau) \frac{K_2(t,\tau) - K_2(t,x)}{\sin\frac{\tau-x}{2}} \mathrm{d}\tau + \bar{Z}(t) K_2(t,x).$$

The kind and properties of the functions $Z(t)$ and $L(t)$ allow us to separate real and imaginary parts in Eq. (36) and to obtain a system of two Fredholm integral equations for the functions $L_1(t)$ and $L_2(t)$ on the interval $[0,\theta_0]$.

4 Results and discussions

Let us investigate the stress and displacement fields near the edge of the spherical cut. It is known that, while calculating these characteristics near the contour of the penny-shaped crack placed on the boundary of the division of two different materials, mathematical difficulties in estimating asymptotically the number of integrals arise. It can be proved that near the boundary circumference of the spherical cut, that is, for $\theta \simeq \theta_0 - \varepsilon$, $(\varepsilon \ll 1)$, the difference of the displacement in the first approximation is represented by the integral

$$\Delta u_\theta + \mathrm{i}\Delta u_r \approx \frac{2r_0}{G_1 G_2} \int_\theta^{\theta_0} \frac{Z(t)\,L(t)\,\mathrm{d}t}{\sqrt{2\cos\theta - 2\cos t}} = \frac{2r_0\sqrt{1-4\gamma^2}}{G_1 G_2} \int_\theta^{\theta_0} \left(\frac{\tan\frac{\theta_0-t}{4}}{\tan\frac{\theta_0+t}{4}}\right)^{\mathrm{i}\lambda} \frac{L(t)\,\mathrm{d}t}{\sqrt{2\cos\theta - 2\cos t}}, \tag{38}$$

where

$$\Delta u_r = u_r^{(1)} - u_r^{(2)} = \frac{r_0}{G_1 G_2} \sum_{n=0}^{\infty} I_n^{(1)} P_n(\cos\theta),$$

$$\Delta u_\theta = u_\theta^{(1)} - u_\theta^{(2)} = \frac{r_0}{G_1 G_2} \sum_{n=0}^{\infty} I_n^{(2)} \frac{n+1/2}{n(n+1)} P_n^{(1)}(\cos\theta).$$

Applying the method of asymptotic integration to the integral (38), where $\theta \sim \theta_0$, we get

$$\Delta u_\theta + \mathrm{i}\Delta u_r \approx \frac{2r_0 L(\theta_0)\sqrt{1-4\gamma^2}}{\cos\frac{1}{2}\theta_0} \left(\frac{\tan\frac{\theta_0-\theta}{4}}{\tan\frac{\theta_0+\theta}{4}}\right)^{\frac{1}{2}+\mathrm{i}\lambda} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1+\mathrm{i}\lambda)}{\Gamma\left(\frac{3}{2}+\mathrm{i}\lambda\right)}, \tag{39}$$

where $\Gamma()$ is the Gamma function. It should be noted that (39) can be derived from (38) by different methods. But, obviously, the most rational one is the substitution

$$\sin\frac{t}{2} = \sin\frac{\theta}{2} + S\left(\sin\frac{\theta_0}{2} - \sin\frac{\theta}{2}\right). \tag{40}$$

The stress fields on the surface of the sphere outside the cut are defined by the right parts of the equalities (12) when $\theta > \theta_0$. An analysis of the series (12) proves that the main stress components for $\theta \approx \theta_0+\varepsilon$ $(\varepsilon \ll 1)$ are found from the sums

$$\sigma_r = \sum_{n=0}^{\infty} \left[a_0 I_n^{(1)} + b_0 I_n^{(2)}\right] \left(n+\frac{1}{2}\right) P_n(\cos\theta),$$

$$\sigma_{r\theta} = \sum_{n=1}^{\infty} \left[\frac{1}{2} b_0 I_n^{(1)} + 2a_0 I_n^{(2)}\right] \frac{\left(n+\frac{1}{2}\right)^2 P_n^1(\cos\theta)}{n(n+1)}. \tag{41}$$

Considering correlations for the Legendre functions [23], asymptotic expressions for the stress fields can be represented in the form

$$\sigma_r \approx -\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left\{\sin\theta\sum_{n=1}^{\infty}\left[a_0 I_n^{(1)} + b_0 I_n^{(2)}\right]\frac{\left(n+\frac{1}{2}\right)P_n^1(\cos\theta)}{n(n+1)}\right\};$$
$$\sigma_{r\theta} \approx \frac{\mathrm{d}}{\mathrm{d}\theta}\sum_{n=0}^{\infty}\left[\frac{1}{2}b_0 I_n^{(1)} + 2a_0 I_n^{(2)}\right]P_n(\cos\theta), \quad (\theta > \theta_0). \tag{42}$$

If $I_n^{(1)}, I_n^{(2)}$ in (42) are substituted by the integral operators (8) and the order of summation and integration is changed, we have

$$\sigma_r \approx \frac{2a_0}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\theta}^{\theta_0}\frac{\sin t\varphi(t)\,\mathrm{d}t}{\sqrt{2\cos t - 2\cos\theta}}, \quad \sigma_{r\theta} \approx 2a_0\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\theta}^{\theta_0}\frac{\psi(t)\,\mathrm{d}t}{\sqrt{2\cos t - 2\cos\theta}}. \tag{43}$$

On the basis of (25), (31), (34) the last relationships, where $\theta \sim \theta_0$, can be presented in the following equivalent form

$$\sigma_{r\theta} + \mathrm{i}\sigma_r \approx -2a_0\sin\theta_0 L(\theta_0)\sqrt{1-4\gamma^2}\int_{\theta}^{\theta_0}\left(\frac{\tan\frac{\theta_0-t}{4}}{\tan\frac{\theta_0+t}{4}}\right)^{\mathrm{i}\lambda}\frac{\mathrm{d}t}{\sqrt{2\cos t - 2\cos\theta}}. \tag{44}$$

After asymptotic integration we will get

$$\sigma_{r\theta} + \mathrm{i}\sigma_r \approx (K_2 + \mathrm{i}K_1)\, r_0^{-0.5}\left(\sin\frac{\theta}{2} - \sin\frac{\theta_0}{2}\right)^{-\frac{1}{2}+\mathrm{i}\lambda}\cdot\left(\sin\frac{\theta}{2} + \sin\frac{\theta_0}{2}\right)^{-\frac{1}{2}-\mathrm{i}\lambda}, \tag{45}$$

where the normal K_1 and tangential K_2 SIF are determined from

$$K_2 + \mathrm{i}K_1 = -4a_0\sqrt{1-4\gamma^2}\sqrt{r_0}L(\theta_0)\cdot\frac{\Gamma(1+\mathrm{i}\lambda)\,\Gamma\left(\frac{1}{2}-\mathrm{i}\lambda\right)}{\Gamma\left(\frac{3}{2}\right)}. \tag{46}$$

The formulas for K_1 and K_2 could be reduced to the results for a penny-shaped crack between dissimilar half-spaces (the problem was studied by Willis [27]) in the limit $r_0 \to \infty$. One of the authors of this paper has carried out such an operation for penny-shaped and spherical cracks in homogeneous space [28]. This demonstrates the reliability of the method presented here. The character of the asymptotic stress field near the boundary circumference of the spherical cut on the interface boundary is the same as near the edge of a penny-shaped crack between two dissimilar half-spaces. If the oscillatory character in this problem is determined by the multipliers $(\sin\theta/2 - \sin\theta_0/2)^{-1/2+\mathrm{i}\lambda}$, then, near the penny-shaped crack, it is determined by the multipliers $(r - r_0)^{-1/2+\mathrm{i}\lambda}$.

We shall now consider the case of an external uniform expansion when the cut surfaces are loaded by normal internal pressure of intensity q. The boundary conditions (2) have the following form

$$\sigma_r^{(1)} = \sigma_r^{(2)}, \quad \sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)}, \quad u_r^{(1)} = u_r^{(2)}, \quad u_\theta^{(1)} = u_\theta^{(2)}, (r = r_0, \theta_0 \le \theta \le \pi),$$
$$\sigma_r^{(1)} = \sigma_r^{(2)} = f_1(\theta) = -q, \quad \sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)} = f_2(\theta) = 0, \quad (r = r_0, 0 \le \theta < \theta_0), \tag{47}$$

The right parts of system (36) are easy to solve. Note that the solution of the system of integral equations (36) depends on the elastic characteristics of the material ν_1, ν_2, G_1, G_2, geometry of the cut r_0, θ_0 and the condition of loading on the surface of the cut. Each of the mentioned parameters has a rather wide range of variation. In this article we will provide conclusions and numerical results which could be of practical value.

First of all, it is worth noting that the dependence of the SIF on Poisson's ratio, having their values in the interval $0{\cdot}2 < \nu_1, \nu_2 < 0{\cdot}42$, is insignificant. For example, with $\nu_1 = \nu_2 = 1/3$ and $\nu_1 = 1/3, \nu_2 = 1/4$ $(G_1 = G_2)$ SIF K_1 and K_2 are, respectively, equal to 0.457 and 0.462.

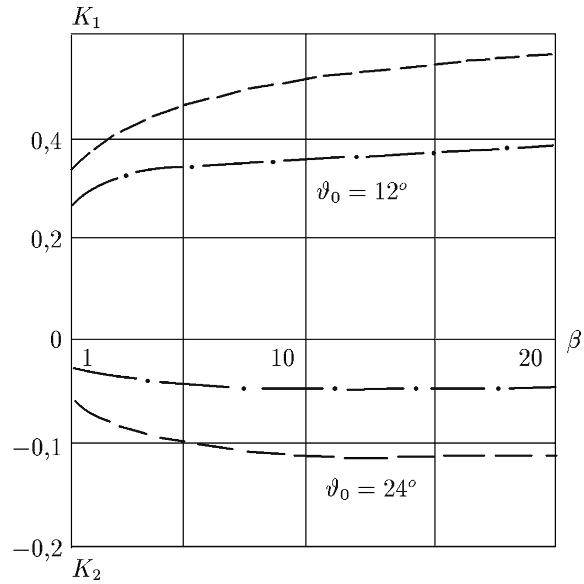

Fig. 3 Dependence of normal K_1 and tangential K_2 SIF on the ratio of shear modules $\beta = G_1/G_2$ inclusion and matrix for different values of the angle of the cut ($\theta_0 = 12; 24$)

Figure 3 shows the SIF behaviour depending on the ratio of shear moduli $\beta = G_1/G_2$ inclusion and matrix for different values of the angle of the cut ($\theta_0 = 12; 24$). Obviously, the main changes of SIF K_1 and K_2 take place when β increases to values of $\beta \gg 5$. This means that a further increase of the ratio of shear moduli of inclusion and matrix should not lead to significant changes in the behaviour of the crack on the interface in composite materials.

These results allow us to compare SIF for composite ($\beta > 1$) and homogeneous ($\beta = 1$) materials. The case $0 < \beta < 1$ is not of practical interest. Figure 3 shows that, with for increased values of β, SIF increase and exceed the corresponding values of SIF for a homogeneous material. If we introduce the parameters $S_1 = K_1^{(c)}/K_1^{(0)}, S_2 = K_2^{(c)}/K_2^{(0)}$, (upper indices correspond to composite and homogeneous materials), then, with $1 < \beta < 60$, the parameters S_i ($i = 1, 2$) are in the interval $1 < S_i < 1{\cdot}5$.

5 Concluding remarks

The class of problems linked to the interaction of matrix and inclusion is interesting from the point of view of the mechanics of composite materials reinforced by hard particles. Cracks on the interface boundary of matrix and inclusion in such materials could appear due to mechanical coercion or environmental influence. Several scientific works present mostly numerical methods for the solution of this class of problems, which is explained by mathematical difficulties arising in the analytical approach. However, only an analytical solution is capable of covering all the singularities of the problem, and giving a general picture of the mechanical conditions of the system dependency on changes in the problem parameters, such as external loading, geometry of the crack, elastic constants of matrix and inclusion, etc., and, thus, to predict the behaviour of the cut when these parameters change. The results of this work show the advantages of such an approach. In particular, we obtained analytical expressions for the components of the stress tensor and the SIF near the edge of the spherical crack on the interface boundary. When the surfaces of the crack are under a normal internal pressure of constant intensity, the dependencies of the SIF on the ratio of the shear moduli β inclusion and the matrix are shown. These dependencies demonstrate that the main changes of the crack's behaviour take place for $\beta \in [1, 5]$. We also compared the values of the SIF for composite and homogeneous materials.

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